JOURNAL OF APPROXIMATION THEORY 58, 121–123 (1989)

An Inequality for Entire Functions of Exponential Type

Q. M. TARIQ

Department of Mathematics, Jamia Millia Islamia, New Delhi, India

Communicated by Oved Shisha

Received March 11, 1986

Using a method due to Lewitan [5], in a form given by Hörmander [4], Frappier [3, Theorem 5] has proved the following result related to the famous inequality of S. Bernstein (see [2, Chap. 11] or [6, Chap. 6]).

THEOREM A. *Let f be an entire function of exponential type τ such that $|f(x)| \leqslant 1$ for $x \in \mathbb{R}$ and $f(0)=0$. Then*

$$|f(x)| \leqslant |\sin \tau x| \qquad \text{for} \quad |x| \leqslant \pi/2\tau. \tag{1}$$

We observe that Theorem A is implicitly contained in a result stated in the above-mentioned paper of Hörmander. According to that result (see the remark following the Corollary on page 26 of [4]) we have

THEOREM B. *Let g be an entire function of exponential type τ such that $g(x)$ is real and $-1 \leqslant g(x) \leqslant 1$ when x is real. If $g(0) = \cos a$, where $0 \leqslant a < \pi$ and $g'(0)=0$, then*

$$g(x) \geqslant \cos(\tau^2 x^2 + a^2)^{1/2} \qquad \text{when} \quad \tau^2 x^2 + a^2 \leqslant \pi^2.$$

Now let f satisfy the conditions of Theorem A and consider the function

$$F(z) = 1 - f(z)\overline{f(\bar{z})}$$

which is of exponential type 2τ with $F(0)=1$, $F'(0)=0$. Since $F(x) \geqslant 0$ for $x \in \mathbb{R}$ we may write (see [1, p. 154] or [2, Sect. 7.5])

$$F(x) = |\varphi(x)|^2$$

where φ is an entire function of exponential type τ such that $\varphi(0)=1$, $\varphi'(0)=0$, and $|\varphi(x)| \leqslant 1$ for $x \in \mathbb{R}$. Thus Theorem B applies with $a=0$ to the function $g(z) = (\varphi(z) + \overline{\varphi(\bar{z})})/2$ and we obtain

$$|\varphi(x)| \geqslant g(x) \geqslant \cos \tau x \qquad \text{for} \quad |x| \leqslant \pi/\tau.$$

0021-9045/89 $3.00

Consequently,

$$1-|f(x)|^2 = F(x) \geqslant \cos^2 \tau x \qquad \text{for} \quad |x| \leqslant \pi/2\tau, \tag{2}$$

which is equivalent to (1). In (2) we have used the fact that $\cos \tau x \geqslant 0$ for $|x| \leqslant \pi/2\tau$.

Note that in our proof of Theorem A we have not required $f(x)$ to be real for real x.

From Theorem B we also deduce the following result whose relevance is abundantly clear.

THEOREM C. *Let f be an entire function of exponential type τ such that $|f(x)| \leqslant 1$ for $x \in \mathbb{R}$. If $|f(0)| = \cos a$, where $0 \leqslant a \leqslant \pi/2$, and $f'(0) = 0$, then*

$$|f(x)| \leqslant \sin\left(\sqrt{(\pi-a)^2 + \tau^2 x^2} - \pi/2\right) \qquad \textit{for} \quad |x| \leqslant \sqrt{a(2\pi - a)}/\tau. \tag{3}$$

Proof. First note that in Theorem B, the hypothesis "$g(0) = \cos a$" may be replaced by "$g(0) \geqslant \cos a$" without any change in the conclusion. In order to prove that (3) holds at an arbitrary point $x_0 \in [-\sqrt{a(2\pi - a)}/\tau, \sqrt{a(2\pi - a)}/\tau]$ we may clearly assume $f(x_0) \neq 0$. Now choose $\gamma \in \mathbb{R}$ such that $f(x_0)\, e^{i\gamma}$ is positive. The function

$$g(z) := -\tfrac{1}{2}\{f(z)\, e^{i\gamma} + \overline{f(\bar{z})}\, e^{-i\gamma}\}$$

is entire and of exponential type τ. It is real and $-1 \leqslant g(x) \leqslant 1$ when x is real. Further, $g(0) \geqslant -\cos a = \cos(\pi - a)$ and $g'(0) = 0$. Hence

$$g(x) \geqslant \cos(\tau^2 x^2 + (\pi - a)^2)^{1/2} \qquad \text{for} \quad |x| \leqslant \sqrt{a(2\pi - a)}/\tau.$$

Since $g(x_0) = -f(x_0)\, e^{i\gamma}$ we obtain

$$|f(x_0)| = f(x_0)\, e^{i\gamma} \leqslant -\cos(\tau^2 x_0^2 + (\pi - a)^2)^{1/2} = \sin\left(\sqrt{(\pi - a)^2 + \tau^2 x_0^2} - \pi/2\right).$$

From Theorems B and C we can easily deduce the following

COROLLARY. *Under the conditions of Theorem C we have*

$$|f''(0)| \leqslant \frac{\sin a}{a}\, \tau^2. \tag{4}$$

In spite of its simplicity, the result contained in this corollary does not seem to have been noted before.

References

1. N. I. ACHIESER, "Theory of Approximation," Frederick Ungar, New York, 1956.
2. R. P. BOAS, JR., "Entire Functions," New York, 1954.
3. C. FRAPPIER, Inequalities for entire functions of exponential type, *Canad. Math. Bull.* **27** (1984), 463–471.
4. L. HÖRMANDER, Some inequalities for functions of exponential type, *Math. Scand.* **3** (1955), 21–27.
5. B. M. LEWITAN, Über eine Verallgemeinerung der Ungleichungen von S. Bernstein und H. Bohr, *Dokl. Akad. Nauk SSSR* **15** (1937), 169–172.
6. Q. I. RAHMAN AND G. SCHMEISSER, "Les inégalités de Markoff et de Bernstein," Séminaire de Mathématiques Supérieures, No. 86 (Été, 1981), Les Presses de l'Université de Montreal, Québec, 1983.